Derivation Of The Capital Asset Pricing Model
Part I - A Single Source Of Uncertainty

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The Capital Asset Pricing Model (CAPM) is used to estimate the required rate of return on an asset. The required rate of return is the rate at which future cash flows produced by the asset are discounted given that asset's relative riskiness. In this white paper we will derive the CAPM equation given that both the individual stock and the market portfolio have a single source of uncertainty. In Part II we will expand the CAPM definition to include an asset that has two sources of uncertainty. An example of such an asset is a minority interest in an entity where the minority shareholder is subject to two sources of risk - (1) the volatility of cash flows associated with the business as a whole and (2) the ability of the controlling shareholders to divert cash flows to themselves or to unproductive ventures either currently or prospectively.

The Market Portfolio Evolution Of Value

The market portfolio is a portfolio consisting of all traded securities that lies on the efficient frontier such that this portfolio has the highest level of return for any given level of risk (i.e. standard deviation). For any given risk-free rate the market portfolio is the only portfolio which can be combined with a risk-free asset to achieve the highest level of return for any given level of risk. In this sense the market portfolio is the optimal portfolio.

We will define the return on the market portfolio to be a function of an expected return (i.e. drift) and an innovation (i.e. an unexpected return). Given this definition of return we can model the return on the market portfolio (M_t) via the following stochastic differential equation (SDE)...

$$\delta M_t = M_t\,\mu_m\,\delta t + M_t\,\sigma_m\,\delta V_t \tag{1}$$

In Equation (1) above $\delta\,M_t$ is the change in the market portfolio's value over the time interval $[t, t + \delta t]$ where t is time in years and δt is an infinitesimal change in time. In that equation μ_m is the expected annual rate of return (i.e. drift), σ_m is annual volatility (i.e. standard deviation) and δV_t is the change in the driving Brownian motion, which is the single source of uncertainty (i.e. risk). The solution to this SDE is...

$$M_t = M_0\,Exp\left\{\left(\mu_m - \frac{1}{2}\sigma_m^2\right)t + \sigma_m\,V_t\right\} \text{ ...where... } V_t \sim N[0,t] \tag{2}$$

Equation (2) defines the value of the market portfolio at time t to be a function of the value of the market portfolio at time zero, the expected annual rate of return, annual volatility, time that has elapsed since time zero and the change in the underlying Brownina motion over the time interval $[0, t]$. We can rewrite Equation (2) (in a sense normalize it) as...

$$M_t = M_0\,Exp\left\{\left(\mu_m - \frac{1}{2}\sigma_m^2\right)t + \sigma_m\,\sqrt{t}\,Y\right\} \text{ ...where... } Y \sim N[0,1] \tag{3}$$

Note that in Equation (3) above we replaced the Brownian motion V_t, which has mean zero and variance equal to elapsed time since time zero, with the product of the square root of time and a normally-distributed random variable with mean zero and variance one.

To make to computations that follow easier to handle we will make the following simplifying definitions...

$$\phi_1 = \left(\mu_m - \frac{1}{2}\sigma_m^2\right)t \text{ ...and... } \phi_2 = \sigma_m\,\sqrt{t} \tag{4}$$

Using the definitions in Equation (4) above we can rewrite Equation (3) as...

$$M_t = M_0 \, Exp\bigg\{\phi_1 + \phi_2 \, Y\bigg\} \;\text{...where...}\; Y \sim N[0,1] \tag{5}$$

We want to deal with log returns so after taking the log of Equation (5) the equation for the log of portfolio value at any time $t > 0$ is...

$$\ln M_t = \ln\left(M_0 \, Exp\bigg\{\phi_1 + \phi_2 \, Y\bigg\}\right) = \ln M_0 + \phi_1 + \phi_2 \, Y \tag{6}$$

Using Equation (6) above the equation for the log return on the market portfolio (R_m) over the time interval $[0,t]$ where $t > 0$ is...

$$\begin{aligned} R_m &= \ln M_t - \ln M_0 \\ &= \ln M_0 + \phi_1 + \phi_2 \, Y - \ln M_0 \\ &= \phi_1 + \phi_2 \, Y \end{aligned} \tag{7}$$

The Individual Stock Evolution Of Value

We will model the return on the individual stock the same way that we modeled the return on the market portfolio. Just as we did in Equation (1) above we can model the return on the individual stock via the following SDE...

$$\delta S_t = S_t \, \mu_s \, \delta t + S_t \, \sigma_s \, \delta W_t \tag{8}$$

Note that in Equation (8) above δW_t is the change in the driving Brownian motion, which is the stock's single source of uncertainty. The Brownian motion W_t may or may not be correlated with the Brownian motion V_t, which is the single source of uncertainty for the market portfolio M_t. Following the format of Equation (2) the solution to the SDE in Equation (8) is...

$$S_t = S_0 \, Exp\bigg\{\left(\mu_s - \frac{1}{2}\sigma_s^2\right) t + \sigma_s \, W_t\bigg\} \;\text{...where...}\; W_t \sim N[0,t] \tag{9}$$

Just as we did in Equation (3) above we can rewrite Equation (9) as...

$$S_t = S_0 \, Exp\bigg\{\left(\mu_s - \frac{1}{2}\sigma_s^2\right) t + \sigma_s \sqrt{t} \, X\bigg\} \;\text{...where...}\; X \sim N[0,1] \tag{10}$$

To make to computations that follow easier to handle we will make the following simplifying definitions...

$$\theta_1 = \left(\mu_s - \frac{1}{2}\sigma_s^2\right) t \;\text{...and...}\; \theta_2 = \sigma_s \sqrt{t} \tag{11}$$

Using the definitions in Equation (11) above we can rewrite Equation (10) as...

$$S_t = S_0 \, Exp\bigg\{\theta_1 + \theta_2 \, X\bigg\} \;\text{...where...}\; X \sim N[0,1] \tag{12}$$

We want to model stock returns as being correlated with market portfolio returns (i.e. systematic risk) and therefore want to correlate the random variable Y in Equation (5) with the random variable X in Equation (12). Given that $\rho_{s,m}$ represents the correlation between the stock and market portfolio returns we will introduce dependence by redefining the normally-distributed random variable X as...

$$X = \rho_{s,m} \, Y + \sqrt{1 - \rho_{s,m}^2} \, X_1 \;\text{...where...}\; Y \sim N[0,1] \;\text{...and...}\; X_1 \sim N[0,1] \tag{13}$$

Note that the random variables Y in Equation (3) above and X_1 in Equation (13) above are independent such that the expected value of the product of these two random variables is...

$$\mathbb{E}\bigg[Y X_1\bigg] = 0 \tag{14}$$

Using Equations (12) and (13) above the equation for stock price at any time $t > 0$ becomes...

$$S_t = S_0\, Exp\bigg\{\theta_1 + \theta_2 \left(\rho_{s,m}\, Y + \sqrt{1-\rho_{s,m}^2}\, X_1\right)\bigg\}$$

$$= S_0\, Exp\bigg\{\theta_1 + \theta_2\, \rho_{s,m}\, Y + \theta_2\, \sqrt{1-\rho_{s,m}^2}\, X_1\bigg\} \tag{15}$$

After taking the log of Equation (15) above the equation for the log of stock price at any time $t > 0$ is...

$$\ln S_t = \ln\left(S_0\, Exp\bigg\{\theta_1 + \theta_2\, \rho_{s,m}\, Y + \theta_2\, \sqrt{1-\rho_{s,m}^2}\, X_1\right)$$

$$= \ln S_0 + \theta_1 + \theta_2\, \rho_{s,m}\, Y + \theta_2\, \sqrt{1-\rho_{s,m}^2}\, X_1 \tag{16}$$

Using Equation (16) above the equation for the log return on the individual stock (R_s) over the time interval $[0, t]$ where $t > 0$ is...

$$R_s = \ln S_t - \ln S_0$$

$$= \ln S_0 + \theta_1 + \theta_2\, \rho_{s,m}\, Y + \theta_2\, \sqrt{1-\rho_{s,m}^2}\, X_1 - \ln S_0$$

$$= \theta_1 + \theta_2\, \rho_{s,m}\, Y + \theta_2\, \sqrt{1-\rho_{s,m}^2}\, X_1 \tag{17}$$

Return Mean, Variance, Covariance and Correlation

Using Appendix Equations (35) and (36) and the definitions from Equation (4) the equations for the mean and variance of market portfolio log returns over the time interval $[0, t]$ are...

$$\text{Mean}_m = \mathbb{E}\bigg[R_m\bigg] = \phi_1 = \left(\mu_m - \frac{1}{2}\sigma_m^2\right) t \tag{18}$$

$$\text{Variance}_m = \mathbb{E}\bigg[R_m^2\bigg] - \left(\mathbb{E}\bigg[R_m\bigg]\right)^2 = \phi_1^2 + \phi_2^2 - \phi_1^2 = \phi_2^2 = \sigma_m^2\, t \tag{19}$$

Using Appendix Equations (37) and (38) and the definitions from Equation (11) the equations for the mean and variance of the individual stock log returns over the time interval $[0, t]$ are...

$$\text{Mean}_s = \mathbb{E}\bigg[R_s\bigg] = \theta_1 = \left(\mu_s - \frac{1}{2}\sigma_s^2\right) t \tag{20}$$

$$\text{Variance}_s = \mathbb{E}\bigg[R_s^2\bigg] - \left(\mathbb{E}\bigg[R_s\bigg]\right)^2 = \theta_1^2 + \theta_2^2 - \theta_1^2 = \theta_2^2 = \sigma_s^2\, t \tag{21}$$

Using Appendix Equations (35), (37) and (39) and the definitions from Equations (4) and (11) the equation for the covariance between market portfolo log returns and the individual stock log returns over the time interval $[0, t]$ is...

$$\text{Covar}_{s,m} = \mathbb{E}\bigg[R_s R_m\bigg] - \mathbb{E}\bigg[R_s\bigg]\mathbb{E}\bigg[R_m\bigg] = \theta_1\, \phi_1 + \theta_2\, \phi_2\, \rho_{s,m} - \theta_1\, \phi_1 = \theta_2\, \phi_2\, \rho_{s,m} = \sigma_s\, \sigma_m\, t\, \rho_{s,m} \tag{22}$$

Using Equation (19), (21) and (22) the correlation of market portfolio log returns and the individual stock log returns over the time interval $[0, t]$ is...

$$\text{Corr}_{s,m} = \frac{\text{Covar}_{s,m}}{\sqrt{\text{Variance}_s}\sqrt{\text{Variance}_m}} = \frac{\sigma_s\, \sigma_m\, t\, \rho_{s,m}}{\sigma_s\, \sqrt{t}\, \sigma_m\, \sqrt{t}} = \rho_{s,m} \tag{23}$$

Note that the correlation coefficient in Equation (23) equals the correlation coefficient in Equation (13) above, which is what we wanted to accomplish all along.

The Market Model

The Market Model is a linear regression where the independent random variable is the log return on the market portfolio and the dependent variable is the log return on the individual stock. The ordinary, least-squares estimation (OLSE) equation for the Market Model is...

$$R_s = \alpha_s + \beta_s\, R_m + \epsilon_s \tag{24}$$

In the market model above R_s is the return on the individual stock as defined by Equation (17), R_m is the return on the market portfolio as defined by Equation (7), α is the regression constant, β_s is the regression coefficient applicable to the independent variable R_m and ϵ_s is the estimation error. Note that the OLSE equation minimizes the squared errors between the estimated value of R_s and the actual (i.e. observed) value of R_s.

Using Equations (19) and (22) the standard regression equation for beta (β_s) in Equation (24) above is...

$$\beta_s = \frac{Cov_{s,m}}{Var_m} = \frac{\theta_2\, \phi_2\, \rho_{s,m}}{\phi_2^2} = \frac{\theta_2}{\phi_2}\, \rho_{s,m} = \frac{\sigma_s \sqrt{t}}{\sigma_m \sqrt{t}}\, \rho_{s,m} = \frac{\sigma_s}{\sigma_m}\, \rho_{s,m} \tag{25}$$

Using Equations (18) and (20) the standard regression equation for alpha (α_s) in Equation (24) above is...

$$\alpha = \text{Mean}_s - \beta_s\, \text{Mean}_m \tag{26}$$

The standard regression mean and variance of the error term ϵ_s in Equation (24) above is...

$$\text{Mean}_\epsilon = \mathbb{E}\Bigg[\epsilon_s\Bigg] = 0 \tag{27}$$

$$\text{Variance}_\epsilon = \mathbb{E}\Bigg[\epsilon_s^2\Bigg] - \Bigg(\mathbb{E}\Bigg[\epsilon_s\Bigg]\Bigg)^2 = \Bigg(1 - \rho_{s,m}^2\Bigg)\, \sigma_s^2 \tag{28}$$

The Capital Asset Pricing Model

If we define R_f to be the risk-free annual rate of return then we can rewrite the Market Model linear regression equation as defined by Equation (24) above as follows...

$$\begin{aligned} R_s &= \alpha_s + \beta_s\, R_m + \epsilon_s \\ &= \alpha_s + \beta_s \Bigg(R_f + R_m - R_f\Bigg) + \epsilon_s \\ &= \alpha_s + \beta_s\, R_f + \beta_s \Bigg(R_m - R_f\Bigg) + \epsilon_s \end{aligned} \tag{29}$$

We can view the above equation as...

$$\text{Compensation for taking on systematic risk} = \beta_s\, (R_m - R_f) \tag{30}$$

$$\text{Compensation for taking on unsystematic risk} = \epsilon_s \tag{31}$$

If the beta coefficient in Equation (29) is equal to zero then either (1) the asset is risk-free ($\sigma_s = 0$) and therefore the asset earns the risk-free rate or (2) the correlation between the asset and the market portfolio is zero ($\rho_{s,m} = 0$) such that all risk can be diversified away and therefore the asset earns the risk-free rate. In either case the required rate of return on this asset is the risk-free rate. If this is the case then we must introduce the following equilibrium constraint...

$$\alpha_s + \beta_s\, R_f = R_f \tag{32}$$

Using Equation (32) above we can rewrite Equation (29) as...

$$R_s = R_f + \beta_s \Bigg(R_m - R_f\Bigg) + \epsilon_s \tag{33}$$

...which is the CAPM equation and completes the derivation.

Appendix

Note that the equations that follow incorporate the following mathematical truths...

$$\mathbb{E}\Bigg[Y\Bigg]=0\ \text{...and...}\ \mathbb{E}\Bigg[Y^2\Bigg]=1\ \text{...and...}\ \mathbb{E}\Bigg[X_1\Bigg]=0\ \text{...and...}\ \mathbb{E}\Bigg[X_1^2\Bigg]=1\ \text{...and...}\ \mathbb{E}\Bigg[Y\,X_1\Bigg]=0 \tag{34}$$

A. The first moment of the distribution of market portfolio log returns is the expected value of the market portfolio log return (R_m) as defined by Equation (7) above. The first moment of the market portfolio log return distribution is...

$$\begin{aligned}
\mathbb{E}\Bigg[R_m\Bigg] &= \mathbb{E}\Bigg[\ln M_t - \ln M_0\Bigg] \\
&= \mathbb{E}\Bigg[\ln M_0 + \phi_1 + \phi_2\,Y - \ln M_0\Bigg] \\
&= \mathbb{E}\Bigg[\phi_1 + \phi_2\,Y\Bigg] \\
&= \phi_1 + \phi_2\,\mathbb{E}\Bigg[Y\Bigg] \\
&= \phi_1
\end{aligned} \tag{35}$$

B. The second moment of the distribution of market portfolio log returns is the expected value of the square of the market portfolio log return (R_m) as defined by Equation (7) above. The second moment of the market portfolio log return distribution is...

$$\begin{aligned}
\mathbb{E}\Bigg[R_m^2\Bigg] &= \mathbb{E}\Bigg[\Bigg(\ln M_t - \ln M_0\Bigg)^2\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg(\phi_1 + \phi_2\,Y\Bigg)^2\Bigg] \\
&= \mathbb{E}\Bigg[\phi_1^2 + 2\,\phi_1\,\phi_2\,Y + \phi_2^2\,Y^2\Bigg] \\
&= \phi_1^2 + 2\,\phi_1\,\phi_2\,\mathbb{E}\Bigg[Y\Bigg] + \phi_2^2\,\mathbb{E}\Bigg[Y^2\Bigg] \\
&= \phi_1^2 + \phi_2^2
\end{aligned} \tag{36}$$

C. The first moment of the distribution of individual stock log returns is the expected value of the individual stock log return (R_s) as defined by Equation (17) above. The first moment of the individual stock log return distribution is...

$$\begin{aligned}
\mathbb{E}\Bigg[R_s\Bigg] &= \mathbb{E}\Bigg[\ln S_t - \ln S_0\Bigg] \\
&= \mathbb{E}\Bigg[\ln S_0 + \theta_1 + \theta_2\,\rho_{s,m}\,Y + \theta_2\,\sqrt{1-\rho_{s,m}^2}\,X_1 - \ln S_0\Bigg] \\
&= \mathbb{E}\Bigg[\theta_1 + \theta_2\,\rho_{s,m}\,Y + \theta_2\,\sqrt{1-\rho_{s,m}^2}\,X_1\Bigg] \\
&= \theta_1 + \theta_2\,\rho_{s,m}\,\mathbb{E}\Bigg[Y\Bigg] + \theta_2\,\sqrt{1-\rho_{s,m}^2}\,\mathbb{E}\Bigg[X_1\Bigg] \\
&= \theta_1
\end{aligned} \tag{37}$$

D. The second moment of the distribution of individual stock log returns is the expected value of the square of the individual stock log return (R_s) as defined by Equation (17) above. The second moment of the individual stock log return distribution is...

$$
\begin{aligned}
\mathbb{E}\Bigg[R_s^2\Bigg] &= \mathbb{E}\Bigg[\Bigg(\ln S_t - \ln S_0\Bigg)^2\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg(\theta_1 + \theta_2\,\rho_{s,m}\,Y + \theta_2\,\sqrt{1-\rho_{s,m}^2}\,X_1\Bigg)^2\Bigg] \\
&= \mathbb{E}\Bigg[\theta_1^2 + \theta_2^2\,\rho_{s,m}^2\,Y^2 + \theta_2^2\Bigg(1-\rho_{s,m}^2\Bigg)X_1^2 + 2\,\theta_1\,\theta_2\,\rho_{s,m}\,Y + 2\,\theta_1\,\theta_2\,\sqrt{1-\rho_{s,m}^2}\,X_1 + 2\,\theta_2^2\,\rho_{s,m}\,\sqrt{1-\rho_{s,m}^2}\,Y\,X_1\Bigg] \\
&= \theta_1^2 + \theta_2^2\,\rho_{s,m}^2\,\mathbb{E}\Bigg[Y^2\Bigg] + \theta_2^2\Bigg(1-\rho_{s,m}^2\Bigg)\mathbb{E}\Bigg[X_1^2\Bigg] + 2\,\theta_1\,\theta_2\,\rho_{s,m}\,\mathbb{E}\Bigg[Y\Bigg] + 2\,\theta_1\,\theta_2\,\sqrt{1-\rho_{s,m}^2}\,\mathbb{E}\Bigg[X_1\Bigg] \\
&\quad + 2\,\theta_2^2\,\rho_{s,m}\,\sqrt{1-\rho_{s,m}^2}\,\mathbb{E}\Bigg[Y\,X_1\Bigg] \\
&= \theta_1^2 + \theta_2^2\,\rho_{s,m}^2 + \theta_2^2\Bigg(1-\rho_{s,m}^2\Bigg) \\
&= \theta_1^2 + \theta_2^2
\end{aligned} \tag{38}
$$

E. The expected value of the product of the individual stock log return (R_s) as defined by Equation (17) above and the market portfolio log return (R_m) as defined by Equation (7) above is...

$$
\begin{aligned}
\mathbb{E}\Bigg[R_s R_m\Bigg] &= \mathbb{E}\Bigg[\Bigg(\ln S_t - \ln S_0\Bigg)\Bigg(\ln M_t - \ln M_0\Bigg)\Bigg] \\
&= \mathbb{E}\Bigg[\Bigg(\theta_1 + \theta_2\,\rho_{s,m}\,Y + \theta_2\,\sqrt{1-\rho_{s,m}^2}\,X_1\Bigg)\Bigg(\phi_1 + \phi_2\,Y\Bigg)\Bigg] \\
&= \mathbb{E}\Bigg[\theta_1\,\phi_1 + \theta_2\,\phi_2\,\rho_{s,m}\,Y^2 + \theta_1\,\phi_2\,Y + \theta_2\,\phi_1\,\rho_{s,m}\,Y + \theta_2\,\phi_1\,\sqrt{1-\rho_{s,m}^2}\,X_1 + \theta_2\,\phi_2\,\sqrt{1-\rho_{s,m}^2}\,X_1\,Y\Bigg] \\
&= \theta_1\,\phi_1 + \theta_2\,\phi_2\,\rho_{s,m}\,\mathbb{E}\Bigg[Y^2\Bigg] + \theta_1\,\phi_2\,\mathbb{E}\Bigg[Y\Bigg] + \theta_2\,\phi_1\,\rho_{s,m}\,\mathbb{E}\Bigg[Y\Bigg] + \theta_2\,\phi_1\,\sqrt{1-\rho_{s,m}^2}\,\mathbb{E}\Bigg[X_1\Bigg] \\
&\quad + \theta_2\,\phi_2\,\sqrt{1-\rho_{s,m}^2}\,\mathbb{E}\Bigg[X_1\,Y\Bigg] \\
&= \theta_1\,\phi_1 + \theta_2\,\phi_2\,\rho_{s,m}
\end{aligned} \tag{39}
$$